COMPARISON OF SOLUTIONS FOR TWO- DIMENSIONAL FLUID FLOW PROBLEMS

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ABSTRACT: *The flow past a wedge of a viscous incompressible fluid has been discussed. The steady flow of Newtonian and Micro-polar fluid is under discussion. A similarity transformation is used to condense the non-linear Partial Differential Equations to a system of ordinary differential equations. The resultant equations are then merged by using the required numerical ways and means. The system is solved numerically. The different three grid sizes are used to maintain the accurateness. The excellent comparison obtained. The results calculated for Newtonian fluid are approximate to the previous. The dimensionless Micro-polar fluid equations are used to establish the resulting governing equation. The difference equations for numerical solutions are finite. At the end all cases with numerical results are described in both graphical and tabular form.*

Keywords: Flow past a wedge, Newtonian fluid, Micro polar fluid

1. INTRODUCTION

We live in an environment of air and of water to a great extent, so that almost everything we do are connected in some way to the science of fluid mechanics [1]. Eringen [2, 3] developed the principle of a class of fluids known as Micro-polar liquids because of limited structure and micro-motion of the fluids apparent certain microscopic effects. The Micro-polar liquids are viscous having additional constants of $\alpha, \beta, \gamma, \lambda, \mu$ and k are viscosity constants in Micro-polar fluids while μ is the viscosity constant in Newtonian fluids.

The effects of the steady rotational motion of polar fluids were studied by Cowen and Pennington. Cowin [4] given the theory of Micro-polar fluids might be applied to electro-hydrological fluids. Guram and Smith [5] studied the geometrical features of steady Ekman flow of Micro-polar fluids. The problem of steady flow in a parallel-walled channel, determined by steady uniform suction through the permeable channel walls for Newtonian fluids was studied by Berman [6].

In the present work, I have scrutinized the steady flow of a Micro-polar fluid past a wedge. Falkner and Skan (1930) [7] studied this problem for the first time for Newtonian fluids and later its solution was studied in detail by Hartee (1937) [8]. They used similarity transformations in order to diminish the Navier-stokes equations [9] to an ordinary differential equation and then gave a series solution of resulting ordinary differential equation. In this work I have considered the steady case of the above problem and determined the numerical solutions for both Micro-polar fluids and Newtonian fluids. The suitable combination of S.O.R. method has been used to integrate these equations numerically

2. MATERIAL AND METHODS

The numerical solution for the Navier Stokes equations for flow past a wedge of a viscous incompressible Newtonian fluid and viscous incompressible Micro-polar fluid have been discussed in this paper. Similarity transformations method is used to condense the Navier stokes equations to one ordinary differential equation. But for two-dimensional Micro-polar flow, it condenses the main equations to two ordinary differential equations. Equations are solved numerically in both cases. Their consequences have been compared and discussed.

The purpose of this study is to found a numerical method, which is more effectual, reasoning, and precise as matched to the other present numerical methods.

2.1 Basic Analysis for two Dimensional Micro-polar Flow Past A Wedge

For incompressible Micro-polar fluids, the main equations given by Eringen [2] are

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{V}) = 0 \tag{1}$$

$$(\lambda + 2\mu + k)\nabla(\nabla.\underline{V}) - (\mu + k)\nabla\times(\nabla\times\underline{V}) + k(\nabla\times\underline{v}) - \nabla \mathrm{P} + \mathrm{P}\,\underline{f} = \mathrm{P}\dot{\underline{V}} \tag{2}$$

$$(a+\beta+y)\nabla(\nabla.\underline{v}) - y(\nabla\times\nabla\times\underline{v}) + k(\nabla\times\underline{V}) - 2k\underline{v} + \mathrm{P}1 = \mathrm{P}j\dot{\underline{v}} \tag{3}$$

From these equations we can have following results

1) Flow is steady, laminar and fully developed.
2) Body forces and body couples are neglected.
3) Co-ordinate system used is the Cartesian system.
4) The flow is two dimensional.
5) The wedge angle β is given by $\beta = \frac{2m}{m+1}$,

   m is a constant.

Let u, v be the constituents of velocity and v_1, v_2, v_3, the micro-rotational components. Under above assumptions the above equations of motion reduce respectively to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

And

$$-(\mu+k)\nabla\times(\nabla\times V) + k(\nabla\times v) - \nabla \mathrm{P} = \mathrm{P}\dot{\underline{V}} \tag{5}$$

$$(a+\beta+y)\nabla(\nabla.\underline{v}) - y(\nabla\times\nabla\times\underline{v}) + k(\nabla\times\underline{V}) - 2k\underline{v} = \mathrm{P}j(\underline{V}.\nabla)\underline{v} \tag{6}$$

In order to solve these equations, we present the following similarity transformations

$$\left.\begin{aligned} u &= u_1 c^m F'(\eta) \\ v &= -\sqrt{\tfrac{m+1}{2} u_1 v^{m-1}}\left(F(\eta) + \frac{m-1}{m+1}\eta F'(\eta)\right) \\ \eta &= -\sqrt{\tfrac{m+1}{2}\frac{u_1}{v}}\, x^{m-1} \end{aligned}\right\} \quad (7)$$

where η is a dimensionless variable.

The particular case considered here is for $m = 1$ and $\beta = 1$.

For this case, the problem under consideration becomes as shown in figure (1).

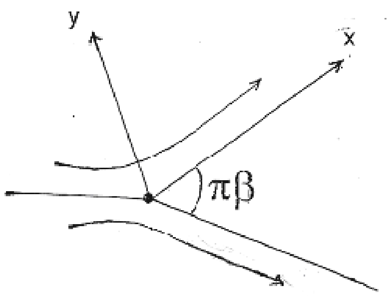


Fig. 1: Physical Model of flow Past a Wedge

The equations (7), therefore, become

$$\left.\begin{aligned} u &= u_1\, x\, F'(\eta) \\ v &= -\sqrt{v\, u_1}\, F(\eta) \\ \eta &= y\sqrt{\frac{u_1}{v}} \end{aligned}\right\} \quad (8)$$

Finally equations of motion to be solved may be written as:

$$\left.\begin{aligned} F''' + F F'' - F'^2 + C_1 H' + 1 &= 0 \\ H'' - C_2 F'' - C_3 H - C_4 (F'H - F H') &= 0 \end{aligned}\right\} \quad (9)$$

Condition to the boundary values:

$$\left.\begin{aligned} &F = 0, \quad F' = 0, \; H = 0 && When\ \eta = 0 \\ &F' = 1, \quad H = 0 && When\ \eta \to \infty \end{aligned}\right\}$$

where $C_2 = \frac{k v}{A\gamma}\sqrt{\frac{u_1}{v}}, \quad C_3 = \frac{2 k v}{u_1\, \gamma}, \quad C_4 = \frac{P\, j\, v}{\gamma}$

are non-dimensional constants.

For numerical purposes, by rewriting the equation (9)

$$P = F'$$

and so equation (9) become as

$$\left.\begin{aligned} P'' + F P' - P^2 + C_1 H' + 1 &= 0 \\ H'' - C_2 P' - C_3 H - C_4 (P H - F H') &= 0 \end{aligned}\right\} \quad (10)$$

While the boundary conditions take the form as:

$$\left.\begin{aligned} &F = 0, \quad P = 0, \; H = 0 && When\ \eta = 0 \\ &P = 1, \quad H = 0 && When\ \eta \to \infty \end{aligned}\right\}$$

If we estimate the derivatives in equation (10) by central – difference approximations at a typical point $\eta = \eta_n$ of the interval $[0, \infty)$, we obtain

$$(2 + hF_n)P_{n+1} - (4 + 2h^2 P_n)P_n + (2 - hF_n)P_{n-1} + C_1 h (H_{n+1} - H_{n-1}) + 2h^2 = 0 \quad (11)$$

$$(2 + C_4 h F_n) H_{n+1} - (4 + 2C_3 h^2 + 2C_4 h^2 P_n) H_n + (2 - C_4\, h F_n) H_{n-1} - C_1\, h (P_{n+1} - P_{n-1}) = 0 \quad (12)$$

Where h denotes the grid length and

$$F(\eta_n) = F_n, P(\eta_n) = P_n, H(\eta_n) = H_n$$

Integrate numerically equation using 1/3Simpson's rule and equations (11) and (12) at each require point of the interval $[0, \infty)$ as follows:

$$F_{n+1} = F_{n-1} + \frac{h}{3}\{ P_{n-1} + 4P_n + P_{n+1} \} \quad (13)$$

With the formula given in [42] to calculate

$$F_2 = F_1 + \frac{h}{24}\{ 9P_1 + 19P_2 - 5P_3 + P_4 \} \quad (14)$$

The system of finite difference equations (13) and (14) are solved by using S.O.R iterative procedure giving

$$P_n = \frac{\omega}{2 + h^2 P_n}\left[\left(1 + \frac{hF_n}{2}\right)P_{n+1} + \left(1 - \frac{hF_n}{2}\right)P_{n-1} - \frac{C_1 h}{2}(H_{n+1} - H_{n-1}) + h^2\right] + (1-\omega)P_n \quad (15)$$

$$H_n = \frac{\omega}{(2 + C_3 h^2 + C_4 h^2 P_n)}\left[\left(1 + \frac{C_4 h}{2}F_n\right)P_{n+1} + \left(1 - \frac{C_4 h}{2}F_n\right)P_{n-1} - \frac{C_2 h}{2}(P_{n+1} - P_{n-1})\right] + (1-\omega)H_n \quad (16)$$

Condition to the appropriate boundary values.

F=0, P=0, H=0 at η=0

P=1, H=0 at η→∞

As ω is a reduction constraint and $1 < \omega < 2$.

The iterative order is as under:

These equations are solved condition to the limits

F=0, P=0, H=0 at η=0

P=1, H=0 at η→∞

Whereas we use these values to establish the matrix of P and H respectively.

The calculated solution for P is then hired into the equations (13) and (14). These equations are solved subject to F = 0, when η=0.

The method is repeated till all the solutions have move toward to some prearranged criteria of accuracy given by

$$\left| f^{n+1}(\infty) - f''(\infty) \right| < 10^{-6}$$

as an ending condition. Also the calculations were checked for changed values of the reduction constraint η between 1 and 2. The best considered value of the reduction constraint is 1.5.

2.2 Basic Analysis for Two Dimensional Newtonian Flow past a Wedge

The Navier-Stokes equations and the continuity equation for incompressible fluid are given by respectively

$$\rho \frac{DV}{Dt} = -\nabla p + \varpi \nabla^2 \underline{V} \quad (17)$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho\, \underline{V}) = 0 \quad (18)$$

Where $\underline{V}$ and ρ are the velocity vector and the density of the fluid. The succeeding suppositions are made for the problem under concern.

- Flow is stable, laminar and fully developed.
- Coordinate system used is the Cartesian system.
- The flow is two dimensional.
- The Wedge angle β is given by $\beta = \frac{2m}{m+1}$,

  m is a constant.

Under the above assumptions, the velocity vector is given by

$$\underline{V} = (u(x, y), \ v(x, y))$$

then the above equations become

$$\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 \quad (19)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + v\frac{\partial^2 u}{\partial y^2} \quad (20)$$

where P is the pressure, $v = \frac{\mu}{\rho}$ is kinematic Viscosity and μ denotes viscosity coefficient. Condition to the limit values

$$\left.\begin{array}{lll} y = 0; & u = 0, & v = 0 \\ y = \infty; & u = U_{(x)} & \end{array}\right\} \quad (21)$$

In case of steady flow the pressure depends only on x, so

$$-\frac{1}{\rho}\frac{dP}{dx} = U\frac{du}{dx}$$

Equation (20) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{du}{dx} + v\frac{\partial^2 u}{\partial y^2} \quad (22)$$

where

$$\left.\begin{array}{l} U(x) = u_1 x^m \\ \frac{du}{dx} = m u_{1,} x^{m-1} \end{array}\right\}$$

To solve equation (19) we present the following similarity transformations

$$\left.\begin{array}{l} u = u_1 x^m F'(\eta) \\ v = -\sqrt{\frac{m+1}{2} u_1 v x^{m-1}}\left[F(\eta)\frac{m-1}{m+1}\right]\frac{v du}{dx}\eta F'(\eta) \\ \eta = \sqrt{\frac{m+1}{2}\frac{u_1}{v}x^{m-1}} \end{array}\right\}$$

where η is a dimensionless variable

$$F''' + FF'' + \beta(1 - F'^2) = 0 \quad (23)$$

Equation (23) is required and the limit values become

$$\left.\begin{array}{ll} F = 0, \quad F' = 0 & \text{when } \eta = 0 \\ F' = 0 & \text{when } \eta \to \infty \end{array}\right\}$$

The equation (23) non-linear and do not give logical solutions. We have to use numerical techniques to solve the equation. For numerical calculation, we rephrase the equations (23) by letting

$$P = F'$$

and so equation (23) become as

$$P'' + FP' + \beta(1 - P^2) = 0 \quad (24)$$

While the boundary condition take the form as:

$$\left.\begin{array}{ll} F = P = 0 & \text{when } \eta = 0 \\ P = 0 & \text{when } \eta \to \infty \end{array}\right\}$$

Two particular cases i.e., for $\beta = 1$, $m = 1$ and $\beta = 0$, $m = 0$, in the problem under consideration are studied. For the above said cases our problem become as shown in figure (1). The equation become as

$$P'' + FP' + (1 - P^2) = 0 \quad (25)$$

$$P'' + FP' = 0 \quad (26)$$

We estimate the derivative in equation (25) and (26) by central – difference approximations at a typical point $\eta = \eta_n$ in the closed- open interval $[0, \infty)$, we got

i) Difference equation for $\beta = 1$, $m = 1$

$$\frac{P_{n+1} - 2P_n + P_{n-1}}{h^2} + F_n\left(\frac{P_{n+1} - P_{n-1}}{2h}\right) + (1 - P_n^2) = 0$$

$$2P_{n+1} - 4P_n + 2P_{n-1} + hF_nP_{n+1} - hF_nP_{n-1} - 2h^2P_n^2 + 2h^2 = 0$$

$$(2 - hF_n)P_{n-1} - (4 + 2h^2P_n)P_n + (2 + hF_n)P_{n+1} + 2h^2 = 0 \quad (27)$$

ii) Difference equation for $\beta = 0$, $m = 0$

$$\frac{P_{n+1} - 2P_n + P_{n-1}}{h^2} + F_n\left(\frac{P_{n+1} - P_{n-1}}{2h}\right) = 0$$

$$2P_{n+1} - 4P_n + 2P_{n-1} + hF_nP_{n+1} - hF_nP_{n-1} = 0$$

$$(2 - hF_n)P_{n-1} - 4P_n + (2 + hF_n)P_{n+1} = 0$$

(28)

where h denotes the grid size and

$$F(\eta_n) = F_n, \; P(\eta_n) = P_n, \; H(\eta_n) = H_n,$$

We now integrate numerically equation using Simpson's $\frac{1}{3}$ rule and (27) and (28) at each require point of the interval $[0, \infty)$ as follows:

$$F_{n+1} = F_{n-1} + \frac{h}{3}\{P_{n-1} + 4P_n + P_{n+1}\}$$

with the formula

$$F_2 = F_1 + \frac{h}{24}\{9P_1 + 19P_2 - 5P_3 + P_4\}$$

This system of fixed – difference equations are solved using S.O.R .

$$P_n = \frac{w}{2 + h^2P_n}\left[\left(1 - \frac{h}{2}F_n\right)P_{n-1} + \left(1 + \frac{h}{2}F_n\right)P_{n+1} + h^2\right] + (1 - \omega)P_n$$

$$P_n = \frac{\omega}{2}\left[\left(1 - \frac{h}{2}F_n\right)P_{n-1} + \left(1 + \frac{h}{2}F_n\right)P_{n+1}\right] + (1 - \omega)P_n$$

depend to the appropriate boundary values

$$\begin{array}{lll} F = 0 & P = 0 & \text{when } \eta = 0 \\ P = 1 & & \text{when } \eta \to \infty \end{array}$$

As 1 < η < 2 and η is a reduction parameter

The iterative order is as follow.

i) These equations are depend on the values

$$\begin{array}{ll} P = 0 & \text{when } \eta = 0 \\ P = 1 & \text{when } \eta \to \infty \end{array}$$

ii) The calculation for *P* is used into equations. These equations are solved for *F* = 0 when η=0 to determine the values of *F*.

The process is continual till the solution has accuracy level given by

$$\left| f^{n+1}(\infty) - f^n(\infty) \right| < 10^{-6}$$

as regarded ending condition. Also the calculation is checked for different values of the reduction constraint η among 1 and 2. The best significance of the reduction constraint is 1.5.

3. RESULTS AND DISCUSSION

The numerical results have been computed for a grid size, h = 0.02 for the three cases.

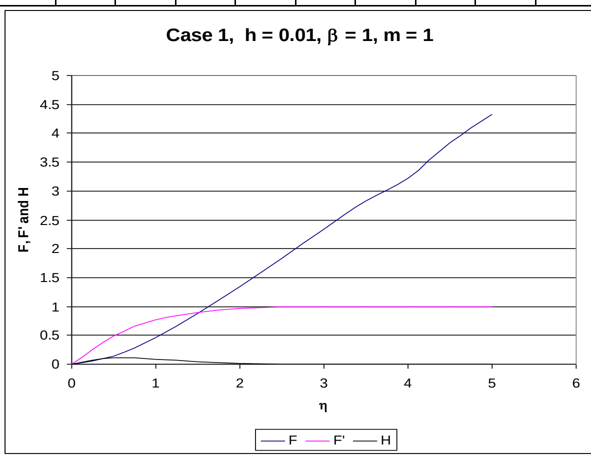
Case I	$C_1 = 0.9$	$C_2 = 0.9$	$C_3 = 1.6$	$C_4 = 1.6$
Case II	$C_1 = 0.49$	$C_2 = 1.26$	$C_3 = 1.74$	$C_4 = 2.24$
Case III	$C_1 = 0.49$	$C_2 = 1.49$	$C_3 = 2.48$	$C_4 = 2.99$

The accuracy of the consequences is checked by comparing on diverse grid sizes. The results are tabulated in (1 – 3), for the values of F, $F' = P$ and H. For $\eta = 3.00$ comparison of the results of Micro-polar fluid for the cases I, II, III with the results of the Newtonian fluid showed in Table 7.

The results are presented for F, $F' = P$ and H in Graphs (1 – 3), for the values of h = 0.01, 0.02 and 0.06 for case I, II, III. For all the grid sizes the value of the reduction parameter is 1.5.

Table-1: Case 1, h = 0.01, m = 1, η = 1

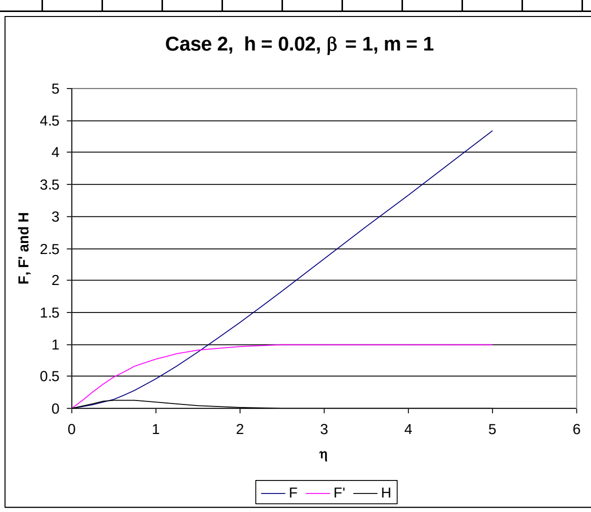
η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.0586	0.2328	0.5148	0.8864	1.3220	1.7951	2.2860	2.7835	3.2830	3.7829
F'(η)	0.0000	0.2341	0.4604	0.6612	0.8165	0.9167	0.9690	0.9908	0.9978	0.9996	1.0000



Graph - 1

Table 2: Case 2, h = 0.02, m = 1, □ = 1,

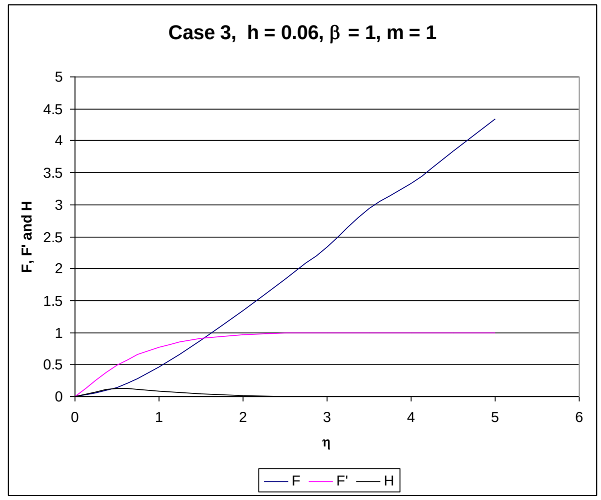
η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.1336	0.4591	0.8871	1.3617	1.8541	2.3515	2.8513	3.3518	3.8518	4.3518
F'(η)	0.0000	0.4945	0.7777	0.9160	0.9731	0.9928	0.9985	0.9997	0.9999	1.0009	1.0000



Graph – 2

Table 3: Case 3, h = 0.06, m = 1, η = 1

η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.0580	0.2307	0.5107	0.8803	1.3143	1.7865	2.2771	2.7744	3.2739	3.7738
F'(η)	0.0000	0.2319	0.4567	0.6569	0.8129	0.9143	0.9679	0.9903	0.9977	0.9996	1.0000



Graph – 3

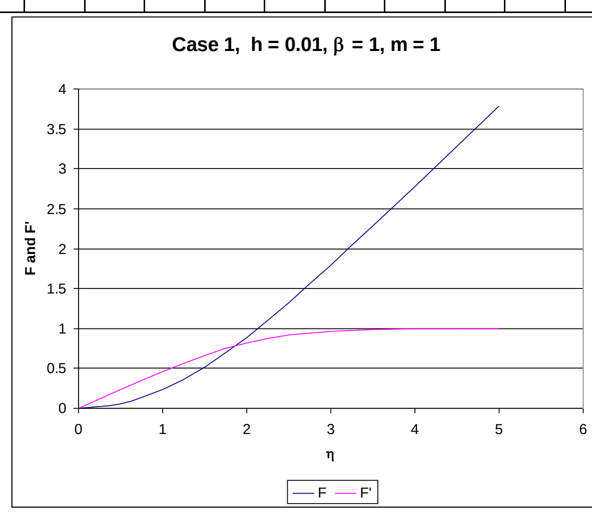
The mathematical solutions for m = 1 and $\beta = 1$ is calculated for three diverse grid sizes, namely, h = 0.01, 0.02 and 0.06 and the result is shown in tables (4 – 6).

The accurateness of the fallouts is checked by equating the fallouts on different grid sizes. The evaluation is brilliant.

The results for F and $F' = P$ are offered at different grid sizes, namely, h = 0.01, 0.02 and 0.06 in graphs (4 – 6). The value of grid size h is shown in tables (4 – 6). For all the grid sizes the value of the reduction parameter is 1.5.

Table 4: Case 1, h = 0.01, m = 1, β = 1

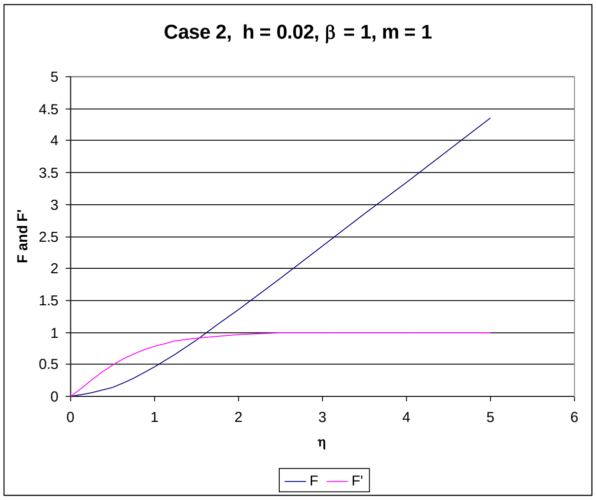
η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.0586	0.2328	0.5148	0.8864	1.3220	1.7951	2.2860	2.7835	3.2830	3.7829
F'(η)	0.0000	0.2341	0.4604	0.6612	0.8165	0.9167	0.9690	0.9908	0.9978	0.9996	1.0000



Graph - 4

Table 5: Case 2, h = 0.02, m = 1, □ = 1

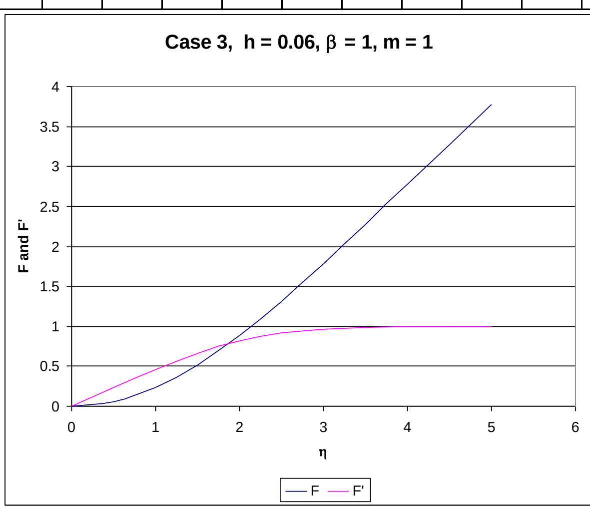
η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.1336	0.4591	0.8871	1.3617	1.8541	2.3515	2.8513	3.3518	3.8518	4.3518
F'(η)	0.0000	0.4945	0.7777	0.9160	0.9731	0.9928	0.9985	0.9997	0.9999	1.0009	1.0000



Graph - 5

Table 6: Case 3, h = 0.06, m = 1, η = 1

η	0.0000	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000
F(η)	0.0000	0.0580	0.2307	0.5107	0.8803	1.3143	1.7865	2.2771	2.7744	3.2739	3.7738
F'(η)	0.0000	0.2319	0.4567	0.6569	0.8129	0.9143	0.9679	0.9903	0.9977	0.9996	1.0000



Graph - 6

Table 7: Evaluation of Results of Micro-polar fluids and Newtonian fluids at η= 3.00, h = 0.06

Case – I	Micro-polar fluid	F = 3.2216	F'= 0.9976
	Newtonian fluid	F = 3.1180	F'= 0.9811
Case – II	Micro-polar fluid	F = 3.4429	F'= 0.8933
	Newtonian fluid	F = 3.3312	F'= 0.9155
Case – III	Micro-polar fluid	F = 3.2918	F'= 0.9872
	Newtonian fluid	F = 3.2988	F'= 0.9233

4. CONCLUSION

In this thesis numerical solution of two dimensional ideal fluid flow problem is presented. The flow past a wedge of a viscous incompressible fluid has been discussed It is a problem of steady flow of Newtonian and Micro-polar fluid. The velocity of the flow field at any point can be obtained by summing the contribution of all the surface elements and adding the contribution of the uniform onset flow. The pressure coefficient at any point of the flow field can be obtained. The comparison of results of two fluids are given above in Table-7.

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